Selection of basic software tools for structured matrix decompositions and perturbations [1]

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Abstract

In this note a survey is given of areas of systems and control where structured matrix problems are important. In identification we mention four different types of data collection : impulse response, input-output pairs, frequency response and covariance data. In each of those, the identification problem can be rewritten in terms of structured matrix problems for which there exist fast decompositions. The use of structured matrix decompositions should yield an improvement in speed of computations. In analysis and design one encounters eigenvalue problems with specific structure such as cyclic, Hamiltonian and symplectic matrices. For those problems it is important to use structure preserving decompositions, mainly to improve the numerical accuracy of the computations, although these algorithms typically yield improved computational complexities as well. We also list the key numerical routines that should be provided in the SLICOT library in order to tackle most of the problems mentioned in this note.

1 Introduction

Structured matrix problems arise in many problems of systems and control. The standard matrix structures that one encounters are Toeplitz and Hankel matrices but in the context of optimization one also often has to solve Hamiltonian, symplectic or patterned eigenvalue and singular value problems.

In the context of systems and control we can discern three areas where these problems enter. The largest area is definitely identification where collected data are typically arranged into a structured matrix before deriving from this the parameters of the model fitting these measurements.

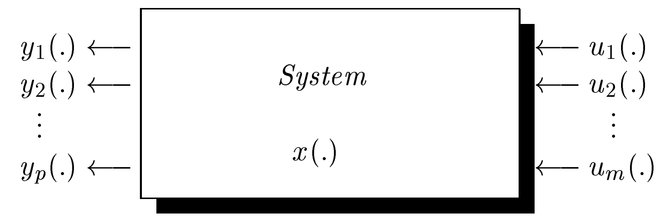
A second area is that of analysis and design of special problems of systems and control. The design of optimal control laws requires e.g. the solution of special eigenvalue problems of Hamiltonian and symplectic type, whereas the analysis of periodic systems requires the solution of "cyclic" eigenvalue problem.

A third area is that of closeness or robustness problems. Here one needs to solve optimization problems with constraints related to some particular structure. This includes structured total least squares problems and structured singular value problems.

2 Structured matrix problems in identification

In this section we survey the identification techniques for linear time invariant systems that typically require the solution of structured matrix problems. Such techniques exist for both continuous-time and discrete-time systems, but in practice the continuous-time case is often solved via the (discrete) sampled versions of the signals anyway. So here we treat the discrete-time case only and we will make a few remarks about the continuous-time case when appropriate. The data that are collected are typically (1) the impulse response of the system, (2) an input output pair of the system, (3) covariance data from the system excited by white noise and finally (4) frequency response measurements of the system.

We consider immediately the multi-input/multi-output (MIMO) case, i.e. we assume a system of the form



where the input vector $u(.)$ is thus m-dimensional and the output vector is p-dimensional.

2.1 Identification via the impulse response

Let us denote the ith unit vector by e_i and consider the response of the input sequence $u_{(j)}(.) = \{e_j,\ 0,\ 0,\ 0, \ldots\}$, i.e., all components of $u_{(j)}(.)$ are zero except the j-th one which is an impulse. Denote the corresponding output vector sequence by $y_{(j)}(.) = \{h_0^{(j)},\ h_1^{(j)},\ h_2^{(j)}, \ldots\}$. Now assemble the p-vectors $h_0^{(j)}$ into the matrix H_0, the p-vectors $h_1^{(j)}$ into the matrix H_1, etc. Then we call the matrix sequence $\{H_0,\ H_1,\ H_2, \ldots\}$ the impulse response of the system $H(z)$. One checks that the (i, j) elements of this matrix sequence is in fact the impulse response of the j-th component of the vector $u(.)$ to the i-th component of the vector $y(.)$, provided all other components of the input vector $u(.)$ are kept zero. As a result of this we have the relation that the matrices H_i are the coefficients of the Taylor expansion of $H(z)$ around z^{-1} :

$$H(z) = H_0 + H_1 z^{-1} + H_2 z^{-2} + H_3 z^{-3} + \ldots$$

If one wants now to identify the system $H(z)$ from this expansion we can try to identify directly a state space model

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k + Du_k, \end{aligned} \tag{1}$$

where the constant matrices A, B, C and D are of dimensions $n \times n$, $n \times p$, $m \times n$, and $m \times p$, respectively. Using the Neumann expansion of $(zI - A)^{-1}$ we obtain :

$$\begin{aligned} H(z) &= D + CBz^{-1} + CABz^{-2} + CA^2Bz^{-3} + CA^3Bz^{-4} + \ldots \\ &= H_0 + H_1 z^{-1} + H_2 z^{-2} + H_3 z^{-3} + H_4 z^{-4} + \ldots \end{aligned}$$

So we have the identities:

$$H_0 = D;\quad H_i = CA^{i-1}B,\ i \geq 1, \tag{2}$$

which can be arranged in an infinite Hankel matrix:

$$\mathcal{H} \doteq \begin{bmatrix} H_1 & H_2 & H_3 & H_4 & \dots \\ H_2 & H_3 & H_4 & \iddots & \dots \\ H_3 & H_4 & \iddots & \iddots & \dots \\ H_4 & \iddots & \iddots & \iddots & \\ \vdots & \vdots & \vdots & & \end{bmatrix} = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \\ \vdots \end{bmatrix} \cdot \begin{bmatrix} B & AB & A^2B & A^3B & \dots \end{bmatrix}. \tag{3}$$

A rank factorization $\mathcal{O}_i\mathcal{C}_j$ of a finite *block Hankel* matrix $\mathcal{H}_{i,j}$, where both i and j are assumed large enough, will yield an observability and controllability matrix from which the matrices A, B and C can be recovered. The number of blocks i and j ought to be chosen such that in

$$\mathcal{H}_{i,j} = \mathcal{O}_i\mathcal{C}_j = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{i-1} \end{bmatrix} \cdot \begin{bmatrix} B & AB & \dots & A^{j-1}B \end{bmatrix}$$

both $\mathcal{O}_i$ and $\mathcal{C}_j$ have full rank n, assuming of course the underlying model is minimal. It is well known from state space theory that the minimum for i is the largest observability index and the minimum for j is the largest controllability index, and both of these are bounded by n. This of course does not prevent to take larger values of i or j, since then the rank of these matrices does not change anymore. From this rank factorization one then defines the submatrices

$$\mathcal{O}_- \doteq \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{i-2} \end{bmatrix}, \ \mathcal{O}_+ \doteq \begin{bmatrix} CA \\ CA^2 \\ \vdots \\ CA^{i-1} \end{bmatrix},$$

$$\mathcal{C}_- \doteq \begin{bmatrix} B & AB & \dots & A^{j-2}B \end{bmatrix}, \ \mathcal{C}_+ \doteq \begin{bmatrix} AB & A^2B & \dots & A^{j-1}B \end{bmatrix}.$$

Since i, j are large enough, these matrices all have full rank n and we can find A from either of the following equations:

$$\mathcal{O}_-A = \mathcal{O}_+, \quad A\mathcal{C}_- = \mathcal{C}_+$$

Both these systems are compatible and hence solvable, yielding

$$A = \mathcal{O}_-^+\mathcal{O}_+ = \mathcal{C}_+\mathcal{C}_-^+.$$

The matrices B and C are found as the first block column and block row of the matrices $\mathcal{C}_j$, respectively $\mathcal{O}_i$. One checks readily that when using the singular value decomposition for this rank factorization, one obtains Gramians $G_o(0, i-1)$ and $G_c(0, j-1)$ over the finite intervals $(0, i-1)$ and $(0, j-1)$ that are equal and diagonal. Such realizations are called balanced and possess a lot of nice properties.

This problem requires rank factorizations of block Hankel matrices that have no special properties such as symmetry or positive definiteness.

2.2 Identification from input-output pairs

If we want to identify a system of order n given in state space coordinates:

$$\left\{ \begin{array}{rcl} x_{k+1} & = & Ax_k + Bu_k \\ y_k & = & Cx_k + Du_k \end{array} \right. \tag{4}$$

explaining the I/O measurements $\{u_i\}$ $\{y_i\}$ for $i = 1, \ldots, N$, where N of course has to be large enough to be able to reconstruct the model $\{A_{nn}, B_{nm}, C_{pn} D_{pm}\}$. As before we have to assume persistence of excitation of the I/O signals. The component-wise identification of the polynomials $n_{ij}(z)/d_{ij}(z)$ of the transfer function between input j and output i leads to an overestimate of the order when assembling these scalar systems into the $p \times m$ transfer function. So identifying scalar models $n_{ij}(z)/d_{ij}(z)$ or $\{A, b_j, c_i, d_{ij}\}$ has to be abandoned in favour of a direct identification of $\{A, B, C, D\}$.

We start by noting that the problem would be much simpler if the sequence of states x_k would be known as well. From

$$\left[\begin{array}{c} x_{k+1} \\ \hline y_k \end{array} \right] = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] \left[\begin{array}{c} x_k \\ \hline u_k \end{array} \right] \tag{5}$$

we can indeed write the concatenated matrix

$$\left[\begin{array}{cccc} x_2 & x_3 & \cdots & x_N \\ y_1 & y_2 & \cdots & y_{N-1} \end{array} \right] = \left[\begin{array}{cc} A & B \\ C & D \end{array} \right] \left[\begin{array}{cccc} x_1 & x_2 & \cdots & x_{N-1} \\ u_1 & u_2 & \cdots & u_{N-1} \end{array} \right]. \tag{6}$$

Under the assumption of persistence of excitation one shows that the right "data matrix" in (6) has full column rank $n + m$ and has thus a right inverse. Equivalently, (6) can be solved in a least squares sense for the evolution matrix

$$\mathcal{E} = \left[\begin{array}{cc} A & B \\ C & D \end{array} \right]. \tag{7}$$

So the problem is solved as soon as the states x_i are determined. Those depend in fact on the choice of coordinates chosen for the state space model : each sequence of states

$$X_{1,N} = [\, x_1 \;\; x_2 \; \ldots \; x_N \,] \tag{8}$$

can only be expected to be known up to an invertible row transformation corresponding to the particular coordinate system of the reconstructed model $\{A, B, C, D\}$. Also the rank condition for (6) to be solvable implies that (7) must be full rank n since this is a submatrix of the right-hand side matrix in (6).

If we define the block Hankel matrices

$$Y_{k,i,j} \doteq \left[\begin{array}{cccc} y_k & y_{k+1} & \cdots & y_{k+j-1} \\ \vdots & \vdots & & \vdots \\ y_{k+i-1} & y_{k+i} & \cdots & y_{k+i+j-2} \end{array} \right] \tag{9}$$

$$U_{k,i,j} \doteq \begin{bmatrix} u_k & u_{k+1} & \cdots & y_{k+j-1} \\ \vdots & \vdots & & \vdots \\ u_{k+i-1} & u_{k+i} & \cdots & y_{k+i+j-2} \end{bmatrix} \tag{10}$$

$$H_{k,i,j} = \left[\frac{Y_{k,i,j}}{U_{k,i,j}}\right], \tag{11}$$

then one can can retrieve a particular sequence of state vectors

$$X_{k+i,j} \doteq [\, x_{k+i}\; x_{k+i+1} \;\; \ldots \;\; x_{k+i+j-1} \,], \tag{12}$$

from the rank condition

$$\text{rank } (H_{k,i,j}) = \text{rank } (U_{k,i,j}) + \text{rank } (X_{k,j}) = mi + n, \tag{13}$$

or equivalently the subspace condition

$$Im\left[X_{k+i,j}^T\right] = Im\left[H_{k,i,j}^T\right] \cap Im\left[H_{k+i,i,j}^T\right] \tag{14}$$

provided the underlying I/O pair is persistently exciting and $i \geq n$, $j \geq (m+p)i$.

Checking this rank condition and computing a sequence of corresponding state vectors $X_{k+i,j}$ requires several QR decompositions of a block Hankel matrix.

2.3 Identification from covariance data

Here we try to identify a system with transfer function $H(z)$ from its response to an input signal which is a stationary white noise process. So the input signal u_k is not known to us, but we know that

$$E\{u_k u_{k-i}\} = \delta_{ik} n$$

where we assume $n = 1$ to start with. Since we do not have access to $\{u_k\}$, we can only try to find a transfer function $h(z)$ such that the output of the above system has a cross correlation

$$E\{y_k y_{k-i}\} = r(k, k-i)$$

which is compatible with the system. If one restricts the system $h(z)$ to be an "all pole" (or autoregressive) system, $h(z) = z^n/d(z$ where $d(z)$ is a certain polynomial of degree n, then this problem turns out to have a simple recursive solution. Write $d(z)/z^n$ as

$$\frac{d(z)}{z^n} = \frac{1 + a_1 z^{-1} + \cdots + a_n z^{-n}}{c}$$

then one tries to find a system of the type

$$h(z) = \frac{c}{a(z^{-1})} = \frac{c}{1 + \sum_{i-1}^{n} a_i z^{-i}}$$

where $a(\cdot)$ is an nth order polynomial. The response to the system is thus:

$$y_k = cu_k - \sum_{i=1}^{n} a_i y_{k-i}. \tag{15}$$

If the a_i and c coefficients are fixed, then $\{y_k\}$ is again a stationary process. Therefore,

$$E\{y_k, y_{k-j}\} = r_j = r_{-j}$$

is only function of the distance i between the samples. Multiplying (15) by y_{k-j} and taking expected values yields

$$r_0 = c - \sum_{i=1}^{n} a_i r_i, \quad r_j = -\sum_{i=1}^{n} a_i r_{j-i}, \quad \text{for } 1 \leq |j| \leq \infty,$$

because we know that u_r is uncorrelated with y_{k-j} for $j > 0$. One can put this together in the matrix equation

$$\begin{bmatrix} r_0 & r_1 & \dots & r_n \\ r_1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & r_1 \\ r_n & \dots & r_1 & r_0 \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} c \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{16}$$

where the matrix T_{n+1} in the left hand side has a Toeplitz structure. In order to solve (16) we thus need a factorization of T_{n+1}. Since it is positive definite and symmetric there exist matrices L and X such that

$$T_{n+1} = LSL^T \tag{17}$$

and

$$X^T T_{n+1} X = S \tag{18}$$

where L is unit lower triangular, $X^T = L^{-1}$ is unit lower triangular and S is diagonal.

There are two fast algorithms for computing the above decomposition. The Levinson algorithm goes back to (1949) and computes X in $0(n^2)$ operations. The Schur algorithm goes back to the previous century and computes L in $0(n^2)$ operations. Both algorithms have been shown recently to be backward stable provided they are implemented with stabilized skew Givens rotations [2] [8].

Vector processes can also be dealt with in the above problem. We then start from an input sequence which is stationary white noise, meaning

$$E\{u_k\, u_{k-i}^T\} = \delta_{ik} I_m.$$

The cross correlation of the output process is measured

$$E\{y_k\, y_{k-i}^T\} = R_i$$

and a predictor polynomial matrix

$$A(z^{-1}) \quad = \quad I + \sum_{i=1}^{n} A_i z^{-i}$$

is found from the decomposition of the block Toeplitz matrix

$$T_{n+1} \quad = \quad \begin{bmatrix} R_0 & R_1 & \ldots & R_n \\ R_1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & R_1 \\ R_n & \ldots & R_1 & R_0 \end{bmatrix}.$$

Both the Levinson and the Schur algorithm have block versions which have complexity n^2m^3 instead of n^3m^3 for a general method for decomposing T_{n+1}.

The routines needed for these decompositions are variants of the Cholesky decomposition of a semidefinite Toeplitz matrix.

2.4 Identification from frequency domain data

Often it is quite easy to excite a system with a periodic signal $u(t)$ and measure the response $y(t)$ of the system to this excitation. This can be done as well in continuous-time as in discrete-time, but for simplicity we will assume here that discrete time signals are being used and that the transfer function to be identified is scalar.

So the transfer function can be assumed to be written as a ratio of two polynomials

$$h(z) = p(z)/d(z),$$

where both are assumed of equal degree n and $d(z)$ is chosen to be monic :

$$\begin{aligned} p(z) &= p_0 + p_1 z + \ldots + p_{n-1} z^{n-1} + p_n z^n \\ d(z) &= d_0 + d_1 z + \ldots + d_{n-1} z^{n-1} + z^n \end{aligned}$$

If we excite this dynamical system with a periodic signal

$$u_k = a_i.z_i^k, \quad z_i = e^{\jmath \omega_i} \Rightarrow |z_i| = 1$$

then the response is again a periodic signal with the same characteristic root, i.e.

$$y_k = b_i.z_i^k = h(z_i).u_k = a_i.h(z_i).z_i^k.$$

If we perform the same experiment with several "frequencies" z_i one finally obtains the constraints

$$b_i d(z_i) = a_i p(z_i), \quad \forall z_i.$$

This can be rewritten as a system of equations by introducing Vandermonde matrices :

$$V_{m,n+1} \doteq \begin{bmatrix} 1 & z_1 & \dots & z_i^n \\ 1 & z_2 & \dots & z_2^n \\ \vdots & & \vdots & \\ 1 & z_m & \dots & z_m^n \end{bmatrix}.$$

The above interpolation constraints can now be rewritten as a "coupled" Vandermonde equation :

$$\begin{bmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_m \end{bmatrix} \begin{bmatrix} 1 & z_1 & \dots & z_i^n \\ 1 & z_2 & \dots & z_2^n \\ \vdots & & & \vdots \\ 1 & z_m & \dots & z_m^n \end{bmatrix} \begin{bmatrix} p_0 \\ \vdots \\ p_{n-1} \\ p_n \end{bmatrix} \tag{19}$$

$$= \begin{bmatrix} b_1 & & & \\ & b_2 & & \\ & & \ddots & \\ & & & b_m \end{bmatrix} \begin{bmatrix} 1 & z_1 & \dots & z_i^n \\ 1 & z_2 & \dots & z_2^n \\ \vdots & & & \vdots \\ 1 & z_m & \dots & z_m^n \end{bmatrix} \begin{bmatrix} d_0 \\ \vdots \\ d_{n-1} \\ 1 \end{bmatrix}. \tag{20}$$

For $m = 2n + 1$ this can be solved exactly and yields the interpolating solution. For $m > 2n + 1$ this problem is over-determined and can be solved in a least-squares sense. This can be incorporated as a basic building block in frequency domain identification techniques but requires additional heuristics to find the correct degree of the model as well as more appropriate least squares measures.

We point out here that fast Vandermonde solvers can be found that have a complexity of $O(mn)$ flops rather that the expected $O(mn^2)$ complexity for standard matrix matrix problems. Notice also that by choosing points z_i equidistantly on the unit circle we can fall back on FFT techniques and hence on a complexity of $m \log m$.

3 Structured matrix problems analysis and design

In this section we look at analysis and design problems involving special structured eigenvalue problems. These occur e.g. in the context of discrete linear time varying systems with periodic coefficients :

$$\left\{ \begin{array}{rcl} E_k x_{k+1} & = & A_k x_k + B_k u_k \\ y_k & = & C_k x_k + D_k u_k, \end{array} \right. \tag{21}$$

where the matrices E_k, A_k, B_k, C_k and D_k vary with period K. The role played by the generalized Schur form for time invariant systems is now replaced by a very similar orthogonal decomposition, called the *periodic Schur form* [1], defined on the pencil

$$\lambda \mathcal{E} - \mathcal{A} \doteq \begin{bmatrix} -A_1 & \lambda E_1 & & \\ & \ddots & \ddots & \\ & & -A_{K-1} & \lambda E_{K-1} \\ \lambda E_K & & & -A_K \end{bmatrix}. \tag{22}$$

Let the $n \times n$ matrices E_k and A_k, $k = 1, \ldots, K$ be such that the pencil $\lambda\mathcal{E} - \mathcal{A}$ is regular. Then there always exist orthogonal transformations Q_k and Z_k, $k = 1, \ldots, K$ such that

$$\operatorname{diag}\{Q_1, \ldots Q_K\}^T (\lambda\mathcal{E} - \mathcal{A}) \operatorname{diag}\{Z_1, \ldots Z_K\} = \begin{bmatrix} -\hat{A}_1 & \lambda\hat{E}_1 & & \\ & \ddots & \ddots & \\ & & -\hat{A}_{K-1} & \lambda\hat{E}_{K-1} \\ \lambda\hat{E}_K & & & -\hat{A}_K \end{bmatrix},$$

where the transformed matrices $\hat{A}_k$ and $\hat{E}_k$ are all upper triangular, except for one matrix – say, $\hat{A}_1$ – which is quasi triangular. The relation with the standard Schur form is that if the E_k matrices are invertible, then the monodromy matrix

$$\hat{\Phi}_{K,1} \doteq \hat{E}_K^{-1}\hat{A}_K \cdots \hat{E}_1^{-1}\hat{A}_1 = Z_1^T(E_K^{-1}A_K \cdots E_1^{-1}A_1)Z_1 = Z_1^T\Phi_{K,1}Z_1.$$

is transformed by the orthogonal similarity Z_1 to its Schur form. Since all matrices are triangular it follows that all transformed monodromy matrices $\hat{\Phi}_{K+k-1,k}$ are quasi triangular as well, and with the same ordering of eigenvalues. This structured matrix decomposition plays a fundamental role in the analysis (and design) of systems with periodic coefficients.

One of the most important design problems for continuous time linear dynamical systems is the solution of the optimal feedback $u = Fx$ minimizing the functional (where we assume $Q \geq 0$, $R > 0$):

$$\begin{aligned} J &= \int_0^\infty \{x^T(t)Qx(t) + u^T(t)Ru(t)\}dt \\ \text{subject to} \quad \dot{x}(t) &= Ax(t) + Bu(t); x(0) = x_0. \end{aligned}$$

If A, B is reachable then the optimal cost will be bounded. Otherwise there always exist an initial state x_0 which can make the cost unbounded if $Q > 0$. Most of the methods that are numerically reliable and of reasonable complexity are based on the state/costate equations

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} = H \begin{bmatrix} x \\ \lambda \end{bmatrix}. \tag{23}$$

This equation is derived from variational principles and its solution with suitable boundary conditions also solves the optimal control problems (see [7] for details).

The corresponding problem for discrete time systems is very similar. Here we want to minimize the functional

$$\begin{aligned} J &= \sum_0^\infty x_k^T Q x_k + u_k^T R u_k \\ \text{subject to} \quad x_{k+1} &= Ax_k + Bu_k, \quad x_0 \text{ given} \end{aligned}$$

Stabilizability is again needed to have a bounded solution for all x. From variational calculus [7], one obtains the equations

$$\begin{bmatrix} x_{k+1} \\ \lambda_r \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ Q & A^T \end{bmatrix} \begin{bmatrix} x_k \\ \lambda_{k+1} \end{bmatrix}$$

or also

$$\left[\begin{array}{cc} I & BR^{-1}B^T \\ 0 & A^T \end{array}\right]\left[\begin{array}{c} x_{r+1} \\ \lambda_{k+1} \end{array}\right] = \left[\begin{array}{cc} A & 0 \\ -Q & I \end{array}\right]\left[\begin{array}{c} x_k \\ \lambda_k \end{array}\right] \tag{24}$$

with boundary conditions specified for x_0 and λ_∞ (see [7] for details).

The underlying eigenvalue problems for these two optimal control problems are the Hamiltonian pencil

$$\lambda\left[\begin{array}{cc} I & 0 \\ 0 & I \end{array}\right] - \left[\begin{array}{cc} A & -BR^{-1}B^T \\ -Q & -A^T \end{array}\right]$$

and the symplectic pencil

$$\lambda\left[\begin{array}{cc} I & BR^{-1}B^T \\ 0 & A^T \end{array}\right] - \left[\begin{array}{cc} A & 0 \\ -Q & I \end{array}\right]$$

Recently developed efficient solutions of these eigenvalue problems in fact make use of the periodic Schur form defined earlier.

4 Structured closeness and robustness problems

The eigenvalue problem is one of the most classical problems in matrix theory, and it is also one of the most useful ones. Eigenvalues show up in the mathematical formulation of several real world problems : they are related, among other things, to wavelengths in optical and physical problems, to time constants in chemical and seismic problems, and to natural frequencies in mechanical and electrical engineering problems. The basic mathematical model common to these applications is that of a time-invariant dynamical system, either continuous-time or discrete-time. If we assume a "state-space" formulation with state vector $x(.)$. input $u(.)$ and output $y(.)$, then the "characteristic frequencies" of the system are nothing but the eigenvalues of the matrix A in the state space model. An important concept in this context is the *stability* of the dynamical system, which imposes that the eigenvalues of A (its *spectrum*) lie in a particular region, say Γ, of the complex plane. In the continuous-time case this is the open left half plane and in the discrete-time case it is the open unit disc.

The basic eigenvalue problem is also classical in the area of numerical methods [11], which gives rise to the study of the *sensitivity* of eigenvalues of a matrix. If one considers a perturbation Δ of the matrix A then the sensitivity of an eigenvalue $\lambda(A)$ is, in a sense, the *derivative* of $\lambda(A+\Delta)$ versus the perturbation Δ. This indicates *how fast* the eigenvalue λ will change with the perturbation Δ. One can use this to *estimate* how large Δ has to be in order a particular eigenvalue $\lambda(A+\Delta)$ to reach the boundary $\partial\Gamma$ of the stable region Γ. This is an important question since it indicates for what values of Δ the matrix $A+\Delta$ tends to lose its stability. One is interested in finding *exact bounds* for Δ so that $\lambda(A+\Delta)$ is guaranteed to remain stable. Because eigenvalues are continuous functions of Δ, this is equivalent to finding the smallest perturbation Δ such that one of the eigenvalues *crosses* the boundary of Γ, i.e. $\lambda(A+\Delta) \in \partial\Gamma$. This is also called a *robustness* problem for the matrix A and clearly can not be solved via a sensitivity analysis, since that only studies infinitesimal perturbations and derivatives of the eigenvalues of A.

Robustness of eigenvalues was never a popular topic in matrix theory, but it received a lot of attention in the dynamical systems area. In the last decade, there has been a significant growth in the theory and techniques of what is called *robust control.* These developments mainly center around two concepts : H_∞-norm and $\mu_{\mathcal{D}}$-norm. They both provide a framework for synthesizing *robust* controllers for linear systems, in the sense that they achieve a desired system performance in the presence of a significant amount of uncertainty in the system. The *transfer matrix* $H(\lambda) = C(\lambda I - A)^{-1}B + D$ of a dynamical system is a rational matrix and its H_∞ norm is then defined as

$$\|H(\lambda)\|_\infty \doteq \sup_{\lambda\in\partial\Gamma} \|H(\lambda)\|_2 \doteq \sup_{\lambda\in\partial\Gamma} \sigma_1[H(\lambda)] \tag{25}$$

where $\sigma_1[M]$ is the largest singular value of M and $\partial\Gamma$ is the imaginary axis for continuous-time systems and the unit circle for discrete-time systems. If one considers perturbations of the type $A + B\Delta C$ in the standard matrix problem, then the perturbation Δ of smallest 2-norm, that destabilizes $A + B\Delta C$, has precisely norm

$$\|\Delta_{min}\|_2 = \left\{ \sup_{\lambda\in\partial\Gamma} \|C(\lambda I - A)^{-1}B\|_2 \right\}^{-1} = \|C(\lambda I - A)^{-1}B\|_\infty^{-1}. \tag{26}$$

The problem becomes more involved when one imposes Δ to be *real.* A very similar bound is then obtained :

$$\|\Delta_{min}\| = \left\{ \sup_{\lambda\in\partial\Gamma} \mu_{\mathcal{R}}[\,H(\lambda)\,] \right\}^{-1}, \tag{27}$$

where, for any $M \in \mathcal{C}^{p\times m}$,

$$\mu_{\mathcal{R}}(M) = \inf_{\gamma\in(0,1]} \sigma_2 \left(\begin{bmatrix} \Re M & -\gamma \Im M \\ \gamma^{-1}\Im M & \Re M \end{bmatrix} \right). \tag{28}$$

Structured singular values [12] have been introduced to provide a perturbation bound for structured uncertainties in control system analysis and design. Therefore, this approach can be viewed as a complement for the H_∞ approach. The $\mu_{\mathcal{D}}$ norm of a stable transfer function $H(\lambda)$ is defined as

$$\|H(\lambda)\|_{\mu_{\mathcal{D}}} \doteq \sup_{\lambda\in\partial\Gamma} \mu_{\mathcal{D}}[H(\lambda)], \tag{29}$$

where, for any $M \in \mathcal{C}^{n\times n}$, the *largest structured singular value* $\mu_{\mathcal{D}}$ is defined as

$$\mu_{\mathcal{D}}(M) = \left\{ \min_{\{\Delta\in\mathcal{D},\, \det(I-\Delta M)=0\}} \sigma_1(\Delta) \right\}^{-1} \tag{30}$$

and $\mathcal{D}$ is the class of block diagonal matrices Δ :

$$\begin{array}{l} \Delta = \mathrm{diag}\{\delta_1^r I_{k_1}, \ldots, \delta_p^r I_{k_p}, \delta_1^c I_{k_{p+1}}, \ldots, \delta_q^c I_{k_{p+q}}, \Delta_1^C, \ldots, \Delta_r^C\}, \\ \delta_i^r \in \mathcal{R},\ \delta_i^c \in \mathcal{C},\ \Delta_i^C \in \mathcal{C}^{c_i\times c_i}. \end{array} \tag{31}$$

In a linear system with multiple independent norm-bounded perturbations, it is always possible by rearranging the system to isolate the perturbations as a single large block diagonal perturbation Δ. The smallest perturbation destabilizing A is again equal to

$$\|\Delta_{min}\| = \|C(\lambda I - A)^{-1}B\|_{\mu_{\mathcal{D}}}^{-1}. \tag{32}$$

This structured singular value problem solves a wide range of problems and can be viewed as the most general formulation of the perturbation problems introduced earlier and which all relate to the *robustness* of the *stability* of a given matrix A.

This problem requires the computation of the structured singular values of a matrix S with respect to certain perturbation structures. This is related to the computation of the singular values of matrices with a given structure. These also need to be solved via optimization techniques in which the basic building blocks are structured matrix decompositions [10].

5 Proposed routines for structured matrix problems

The different sections in this working note suggest the elaboration of the following routines :

- A routine for computing the Levinson decomposition
$$T = L^{-1}DL^{-T}$$
of a semidefinite (block) Toeplitz matrix (with rank determination)
- A routine for computing the Schur decomposition
$$T = LDL^T$$
of a semidefinite (block) Toeplitz matrix (with rank determination)
- A routine for computing the QR decomposition
$$T = QR$$
of a (block) Toeplitz or Hankel matrix (with rank determination)
- A routine for constructing a (Gohberg-Semencul like) formula for the solution or least square solution of a (scalar) Toeplitz matrix
$$G = f(T)$$
- A routine for solving a Vandermonde system of equations
$$Vx = b$$
- A routine for solving the interpolation or least squares approximation to a (scalar) rational interpolation problem
- A routine for solving the periodic Schur form of a sequence of K real or complex matrices
- A routine for computing the complex and real stability radius of a perturbed matrix
$$A + B\Delta C$$
- A routine for computing a bound for the structured singular value of a perturbed matrix
$$A + B\Delta C$$
- (optional) A routine for computing the block LDL^T decomposition of a scalar Hankel matrix using lookahead techniques.

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