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An Introduction to $\mathcal{H}_\infty$ Optimisation Designs [1]

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Abstract

This *Niconet* report is prepared for users of the software package *SLICOT* who are not familiar with the $\mathcal{H}_\infty$ optimisation design approach. Together with the *Niconet* reports [1, 2, 3], it is hoped that the reader would have a general idea about the $\mathcal{H}_\infty$ method, know how to use the algorithms availabe in *SLICOT* to synthesize a controller for a standard $\mathcal{H}_\infty$ optimisation problem and, furthermore, be aware of some difficulties such as singularity in the $\mathcal{H}_\infty$ controllers design.

Key Words: $\mathcal{H}_\infty$ optimisation methods, SLICOT

1 Introduction

It may be fair to say that the robustness is the most important issue in the control systems design, simply due to the inevitable existence of uncertainties in the real environment where the control systems are in operation. The purposes for the feedback control mechanism are to stabilize the plant, if it is unstable, and to reduce the effects of plant dynamic perturbations and to attenuate system responses to external disturbances/noises.

Though always appreciated, the need and importance of robustness in control systems design has been brought into the limelight during the last two decades. In classical single-input single-output control, robustness is achieved by ensuring good gain and phase margins. Designing for good stability margins usually also results in good, well-damped time responses, i.e. good performance. When multivariable design techniques were first developed in the 1960s, the emphasis was laid on achieving good performance, and not on robustness. Those multivariable rechniques were based on linear quadratic performance criteria and Gaussian disturbances, and proved to be successful in many aerospace applications where accurate mathematical models can be obtained, and descriptions for external disturbances/noise based on white noise are considered appropriate. However, application of such methods, commonly referred to as the linear quadratic Gaussian (LQG) methods, to other industrial problems made apparent the poor robustness properties exhibited by LQG controllers. This led to a substantial research effort to develop a theory which could explicitly address the robustness issue in feedback design. The pioneering work in the development of the forthcoming theory, now known as the $\mathcal{H}_\infty$ control theory, was conducted in the early 1980s by Zames [4] and Zames & Francis [5]. In $\mathcal{H}_\infty$ approach, the designer from the outsets, specifies a model of system uncertainty, such as additive perturbation and/or output disturbance, that is most suited to the problem at hand. A constrained optimisation is then performed to maximise the robust stability of the closed-loop system to the type of uncertainty chosen, the constraint being the internal stability of the feedback system. In most cases, it would be sufficient to seek a feasible controller such that the closed-loop system achieves certain robust stability. Performance objectives can also be included in the optimisation cost function. Elegant solution formulae have been developed, which are based on the solutions of two algebraic Riccati equations, and are readily available in software packages such as *SLICOT* and *Matlab*. In general, the standard $\mathcal{H}_\infty$ optimisation formulation, which is discussed in this report, would achieve robust stability and good nominal performance for a linear, time-invariant, finite-dimension control system.

This *Niconet* report is prepared for users of the software package *SLICOT* who are not familiar with the $\mathcal{H}_\infty$ optimisation design approach. Together with the *Niconet* reports [1, 2, 3], it is hoped that the reader would have a general idea about the $\mathcal{H}_\infty$ method, know how to use the algorithms availabe in *SLICOT* to synthesize a controller for a standard $\mathcal{H}_\infty$ optimisation problem and, furthermore, be aware of some difficulties such as singularity in the $\mathcal{H}_\infty$ controllers design. Detailed derivations will not be presented in the report. Interested readers are referred to several books such as [6, 7, 8] and the references therein.

2 Signals and Systems

In this section the basic concepts concerning signals and systems are to be reviewed in brief. A control system interacts with its environment through command signals, disturbance signals and noise signals, etc. Tracking error signals and actuator driving signals are also important in control systems design. For the purpose of analysis and design, appropriate measures, the norms, must be defined for describing the size of these signals. From the signal norms, we can then define induced norms to measure the "gain" of the operator which represents the control system.

2.1 Vector norms and signal norms

Let the linear space X be $\mathcal{F}^m$, where $\mathcal{F} = \mathcal{R}$ for the field of real numbers, or $\mathcal{F} = \mathcal{C}$ for complex numbers. For $x = [x_1, x_2, ..., x_m]^T \in X$, the p-norm of the vector x is defined by

1-norm	$\|x\|_1$	$:=$	$\sum_{i=1}^m \|x_i\|$,	for $p = 1$
p-norm	$\|x\|_p$	$:=$	$(\sum_{i=1}^m \|x_i\|^p)^{1/p}$,	for $1 < p < \infty$
∞-norm	$\|x\|_\infty$	$:=$	$\max_{1\le i\le m} \|x_i\|$,	for $p = \infty$.

When $p = 2$, $\|x\|_2$ is the familiar Euclidean norm.

When X is a linear space of continuous or piecewise continuous time scalar-valued signals $x(t)$, $t \in \mathcal{R}$, the p-norm of a signal $x(t)$ is defined by

1-norm	$\|x\|_1$	$:=$	$\int_{-\infty}^{\infty} \|x(t)\|dt$,	for $p = 1$
p-norm	$\|x\|_p$	$:=$	$\left(\int_{-\infty}^{\infty} \|x(t)\|^p dt\right)^{1/p}$,	for $1 < p < \infty$
∞-norm	$\|x\|_\infty$	$:=$	$\sup_{t\in\mathcal{R}} \|x(t)\|$,	for $p = \infty$.

The normed spaces, consisting of signals with finite norm as defined correspondingly, are call $L^1(\mathcal{R})$, $L^p(\mathcal{R})$ and $L^\infty(\mathcal{R})$, respectively. From a signal point of view, the 1-norm, $\|x\|_1$ of the signal $x(t)$ is the integral of its absolute value. The square of the 2-norm, $\|x\|_2^2$, is often called the *energy* of the signal $x(t)$ since that is what it is when $x(t)$ is the current through a 1Ω resistor. The ∞-norm, $\|x\|_\infty$, is the amplitude or peak value of the signal, and the signal is bounded in magnitude if $x(t) \in L^\infty(\mathcal{R})$.

When X is a linear space of continuous or piecewise continuous *vector-valued* functions of the form $x(t) = [x_1(t), x_2(t), \cdots, x_m(t)]^T$, $t \in \mathcal{R}$, we may have

$$L^p_m(\mathcal{R}) \quad := \quad \{x(t): \quad \|x\|_p \quad = \left(\int_{-\infty}^{\infty} \sum_{i=1}^m |x(t)|^p dt\right)^{1/p} < \infty \text{ , for } 1 \le p < \infty\}$$

$$L^\infty_m(\mathcal{R}) \quad := \quad \{x(t): \quad \|x\|_\infty \quad = \sup_{t\in\mathcal{R}} \|x(t)\|_\infty < \infty\}$$

Some signals are useful for control systems analysis and design, for example, the sinusoidal signal, $x(t) = A\sin(\omega t + \phi)$, $t \in \mathcal{R}$. It is unfortunately not a 2-norm signal because of the infinite energy contained. However, the average power of $x(t)$

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x^2(t)dt$$

exists. The signal $x(t)$ will be called a *power signal* if the above limit exists. The square root of the limit is the well-known r.m.s. (*root-mean-square*) value of $x(t)$. It should be noticed that the average power does not introduce a norm, since a nonzero signal may have zero average power.

2.2 System norms

System norms are actually the input-output gains of the system. Suppose that $\mathcal{G}$ is a linear and bounded system which maps the input signal $u(t)$ into the output signal $y(t)$, where $u \in (U, \|\cdot\|_U)$, $y \in (Y, \|\cdot\|_Y)$. U and Y are the signal spaces, endowed with the norms $\|\cdot\|_U$ and $\|\cdot\|_Y$, respectively. Then the norm, maximum system gain, of $\mathcal{G}$ is defined as

$$\|\mathcal{G}\| := \sup_{u\neq 0} \frac{\|\mathcal{G}u\|_Y}{\|u\|_U} \tag{2.1}$$

or

$$\|\mathcal{G}\| = \sup_{\|u\|_U=1} \|\mathcal{G}u\|_Y = \sup_{\|u\|_U\le 1} \|\mathcal{G}u\|_Y.$$

Obviously, we have

$$\|\mathcal{G}u\|_Y \le \|\mathcal{G}\| \cdot \|u\|_U.$$

If $\mathcal{G}_1$ and $\mathcal{G}_2$ are two linear, bounded and compatible systems, then

$$\|\mathcal{G}_1\mathcal{G}_2\| \le \|\mathcal{G}_1\| \cdot \|\mathcal{G}_2\|.$$

$\|\mathcal{G}\|$ is called the *induced norm* of $\mathcal{G}$ with regard to the signal norms $\|\cdot\|_U$ and $\|\cdot\|_Y$. In this report, we are particularly interested in the so called ∞-norm of a system. For a linear, time-invariant, stable system $\mathcal{G}$: $L^2_m(\mathcal{R}) \to L^2_p(\mathcal{R})$, the ∞-norm, or the induced 2-norm, of $\mathcal{G}$ is given by

$$\|\mathcal{G}\|_\infty = \sup_{\omega \in \mathcal{R}} \|G(j\omega)\|_2 \tag{2.2}$$

where $\|G(j\omega)\|_2$ is the spectral norm of the $p \times m$ matrix $G(j\omega)$ and $G(s)$ is the transfer function matrix of $\mathcal{G}$. Hence, the ∞-norm of a system describes the maximum energy gain of the system and is decided by the peak value of the largest singular value of the frequency response matrix over the whole frequency axis. This norm is called the $\mathcal{H}_\infty$-norm, since we denote by $\mathcal{H}_\infty$ the linear space of all stable linear systems.

2.3 Uncertainties

It is well understood that the uncertainty is unavoidable in a real control system. The uncertainty can be classified into two categories: disturbance signals and dynamic perturbations. The former includes input or output disturbance (such as gust to an aircraft), sensor noise and actuator noise, etc. The latter represents the discrepancy between the mathematical model and the actual dynamics of the system in operation. A mathematical model of any real system is always just an approximation of the true system dynamics. Typical sources of the discrepancy include unmodelled (usually high frequency) dynamics, neglected nonlinearities in the modelling, effects of deliberate reduced order models, and system parameter variations due to environmental changes and torn-and-worn factors.

In the $\mathcal{H}_\infty$ design, the dynamic perturbations which may occur in different parts of a system are lumped into one single perturbation Δ. This uncertainty representation is referred to as "unstructured" uncertainty. The block Δ is uncertain, but norm-bounded, say $\bar{\sigma}[\Delta(j\omega)] \leq \delta(\omega)$, for all frequency ω, where δ is a known scalar function and $\bar{\sigma}[M]$ denotes the largest singular value of a matrix M. The unstructured dynamics uncertainty in a control system can be described in different ways, such as listed in the following, where $G_p(s)$ denotes the actual, perturbed system dynamics and $G_o(s)$ the nominal model of the system.

1. Additive perturbation:

$$G_p(s) = G_o(s) + \Delta(s) \tag{2.3}$$

2. Inverse additive perturbation:

$$(G_p(s))^{-1} = (G_o(s))^{-1} + \Delta(s) \tag{2.4}$$

3. Input multiplicative perturbation:

$$G_p(s) = G_o(s)[I + \Delta(s)] \tag{2.5}$$

4. Output multiplicative perturbation:

$$G_p(s) = [I + \Delta(s)]G_o(s) \tag{2.6}$$

5. Inverse input multiplicative perturbation:

$$(G_p(s))^{-1} = [I + \Delta(s)](G_o(s))^{-1} \tag{2.7}$$

6. Inverse output multiplicative perturbation:

$$(G_p(s))^{-1} = (G_o(s))^{-1}[I + \Delta(s)] \tag{2.8}$$

7. Left coprime factor perturbations:

$$G_p(s) = (\tilde{M} + \Delta_{\tilde{M}})^{-1}(\tilde{N} + \Delta_{\tilde{N}}) \tag{2.9}$$

8. Right coprime factor perturbations:

$$G_p(s) = (N + \Delta_N)(M + \Delta_M)^{-1} \tag{2.10}$$

In the last two representations, $(\tilde{M}, \tilde{N})/(M, N)$ are the left/right coprime factorizations of the nominal system model $G_o(s)$, respectively; and $(\Delta_{\tilde{M}}, \Delta_{\tilde{N}})\ /(\Delta_M, \Delta_N)$ are the perturbations on the corresponding factors [9].

It should be noted that a successful $\mathcal{H}_\infty$ design would depend on, to certain extent, an appropriate description of the perturbation considered, though theoretically most representations are interchangeable.

3 $\mathcal{H}_\infty$ Optimisation Approaches

A control system is *robust* if it remains stable and achieves certain performance criteria in the presence of possible uncertainties as discussed in §2.3. The robust design is to find a controller, for a given system, such that the closed-loop system is robust. The $\mathcal{H}_\infty$ optimisation approach, being developed in the last two decades and still an active research area, has been shown to be an effective and efficient robust design method for linear, time-invariant control systems. We will first introduce in this section the *Small Gain Theorem*, which has been used widely in the $\mathcal{H}_\infty$ optimisation methods, and then discuss the nominal performance and robust stabilisation designs using the $\mathcal{H}_\infty$ optimisation ideas.

3.1 Small Gain theorem

The small gain theorem is of central importance in the derivation of many stability tests. In general, it provides only a sufficient condition for stability and is therefore potentially conservative. The small gain theorem is applicable to general operators. What will be included here is however a version which is suitable for the $\mathcal{H}_\infty$ optimisation designs.

Consider the feedback configuration in Figure 1, where $G_1(s)$ and $G_2(s)$ are the transfer function matrices of corresponding linear, time-invariant systems. We then have the following theorem.

Theorem 1 [10] *If $G_1(s)$ and $G_2(s)$ are stable, i.e., $G_1 \in \mathcal{H}_\infty$, $G_2 \in \mathcal{H}_\infty$, then*

$$\|G_1 G_2\|_\infty < 1 \ \text{ implies } \ \text{the closed} - \text{loop system is internally stable}$$

■

3.2 Performance considerations

Figure 2 depicts a typical closed-loop system configuration, where G is the plant and K the controller to be designed. r, y, u, e, d, n are, respectively, the reference input, output, control signal, error signal, disturbance and measurement noise. With a little abuse of notations, we do not distinguish the notations of signals in time- or frequency-domain. The following relationships are immediately available.

$$y = (I + GK)^{-1}GKr + (I + GK)^{-1}d - (I + GK)^{-1}GKn$$

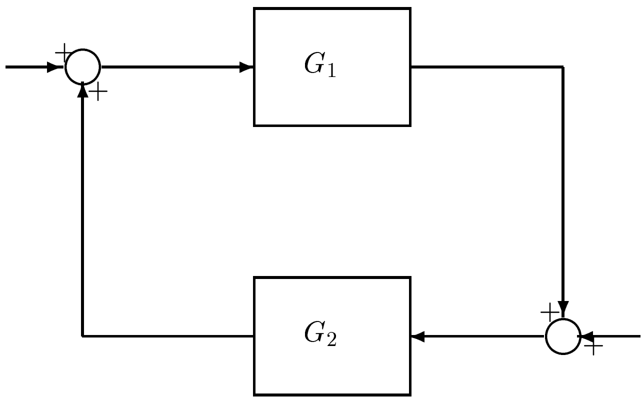

Figure 1: A feedback configuration

$$
\begin{aligned}
u &= K(I+GK)^{-1}r \;-\; K(I+GK)^{-1}d \;-\; K(I+GK)^{-1}n \\
e &= (I+GK)^{-1}r \;-\; (I+GK)^{-1}d \;-\; (I+GK)^{-1}n
\end{aligned}
$$

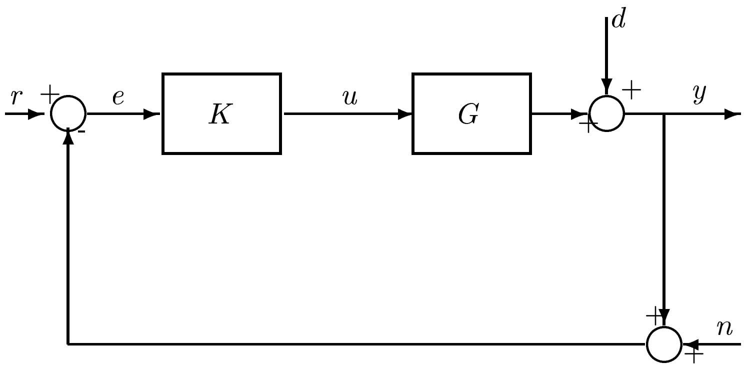

Figure 2: A closed-loop configuration of G and K

Assume that the signals $r,\ d,\ n$ are energy bounded and have been normalized, i.e. lying in the unit ball of L^2 space. We, however, do not know what exactly those signals are. It is required that the usual performance specifications, such as tracking, disturbance attenuation and noise rejection, should be as good as possible for any $r,\ d$ or n whose energy does not exceed 1. From the discussions in Section 2 on signal and system norms, it is clear that we should minimise the ∞-norm, the gain, of corresponding transfer function matrix. Hence, the design problem is that over all stabilizing controller K's, (i.e., those K's make the closed-loop system internally stable), find the optimal one which minimises

- for good tracking,
  $\|(I+GK)^{-1}\|_\infty$;
- for good disturbance attenuation,
  $\|(I+GK)^{-1}\|_\infty$;
- for good noise rejection,
  $\|-(I+GK)^{-1}GK\|_\infty$;
- for less control energy,
  $\|K(I+GK)^{-1}\|_\infty$;

It is conventional to denote $\mathcal{S} := (I+GK)^{-1}$, the *sensitivity* function, and $\mathcal{T} := (I+GK)^{-1}GK$, the *complementary sensitivity* function.

In general, weighting functions would be used in the above minimisation to meet the design specifications. For instance, instead of minimising the sensitivity function alone, we would aim at solving

$$\min_{K\,stabilizing} \|W_1 \mathcal{S} W_d\|_\infty$$

where W_1 is chosen to tailor the tracking requirement and is usually a high-gain low-pass filter type, W_d can be regarded as generator which characterizes all relevant disturbances in the case considered. Usually, the weighting functions are stable and of minimum-phase.

3.3 Robust stabilization

A closed-loop system of the plant G and controller K is robustly stable if it remains stable for all possible, under certain definition, perturbations on the plant. It implies of course that K is a stabilizing controller for the nominal plant G, since we always assume that the perturbation set includes zero (no perturbation). Let us consider the case of additive perturbation as depicted in Figure 3, where $\Delta(s)$ is the perturbation, unknown but stable.

It is easy to work out that the transfer function from the signal v to u is

$$T_{uv} = -K(I+GK)^{-1} = -K\mathcal{S}.$$

As mentioned earlier, the controller K should stabilize the nominal plant G. Hence, from the Small Gain theorem, the closed-loop system is stable if the following holds

$$\|\Delta K(I+GK)^{-1}\|_\infty < 1$$

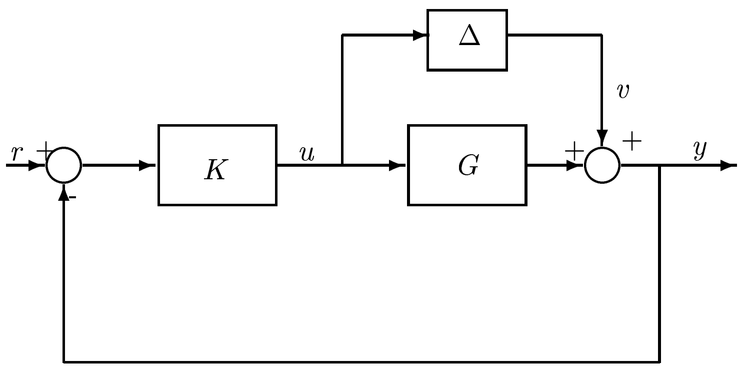

Figure 3: Additive perturbation configuration

or, in a strengthened form,

$$\|K(I+GK)^{-1}\|_\infty < \frac{1}{\|\Delta\|_\infty}.$$

The last condition becomes necessary, when the unknown Δ may have all phases.

If required to find a controller to robustly stabilize the largest possible set of perturbations, in the sense of ∞-norm, it is then clear that we need to solve the following minimisation problem

$$\min_{K stabilizing} \|K(I+GK)^{-1}\|_\infty. \tag{3.1}$$

In many cases, we may have a priori knowledge of the perturbation, say,

$$\bar{\sigma}(\Delta(j\omega)) \leq \bar{\sigma}(W_2(j\omega)) \quad \text{,for all} \quad \omega \in \mathcal{R}.$$

Then, we may rewrite the perturbation block as

$$\Delta(s) = \widetilde{\Delta}(s)W_2(s)$$

where $\widetilde{\Delta}(s)$ is the unit norm perturbation set. Correspondingly, the robust stabilization condition becomes

$$\|W_2K(I+GK)^{-1}\|_\infty < 1$$

and the optimisation problem

$$\min_{K stabilizing} \|W_2K(I+GK)^{-1}\|_\infty. \tag{3.2}$$

Robust stabilization conditions can be derived similarly for other perturbation representations discussed in §2.3.

In the above discussion, the stability of the perturbation has been assumed. Actually, the conclusions are also true, if the perturbed systems have the same number of closed right half plane poles as the nominal system does. If even this is not satisfied, then it will have to use the coprime factor perturbation models in (2.9) or (2.10).

3.4 Standard $\mathcal{H}_\infty$ optimisation configuration

It should be pointed out that, in any real design, it will never be appropriate to use just a single cost function as in (3.1). A reasonable design would use a combination of those functions. For instance, it makes sense to require a good tracking as well as to limit the control signal energy. We may then want to solve the following *mixed sensitivity* problem,

$$\min_{K stabilizing} \left\| \begin{array}{c} (I+GK)^{-1} \\ K(I+GK)^{-1} \end{array} \right\|_\infty . \tag{3.3}$$

This cost function can also be interpreted as the design objectives of nominal performance, the good tracking or disturbance attenuation, and robust stabilization, with regard to additive perturbation.

The $\mathcal{H}_\infty$ optimisation problems can always be formulated into a standard configuration by specifying/grouping signals into sets of external inputs, outputs, input to the controller and output from the controller which of course is the control signal. Such a configuration is depicted in Figure 4. Note that in Figure 4 all the external inputs are denoted by w, z denotes the output signals to be minimised/penalized which includes both performance and robustness measures, y is the vector of measurements available to the controller K and u the vector of control signals. $P(s)$ is called the *generalized* plant or *interconnected* system. The objective is to find a stabilizing controller K to minimise the output z, in the sense of energy, over all w with energy less than or equal to 1. Thus, it is equivalent to minimise the $\mathcal{H}_\infty$-norm of the transfer function from w to z.

Partitioning the interconnected system P as:

$$P(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix}$$

it can be obtained directly

$$z = [P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}]w$$

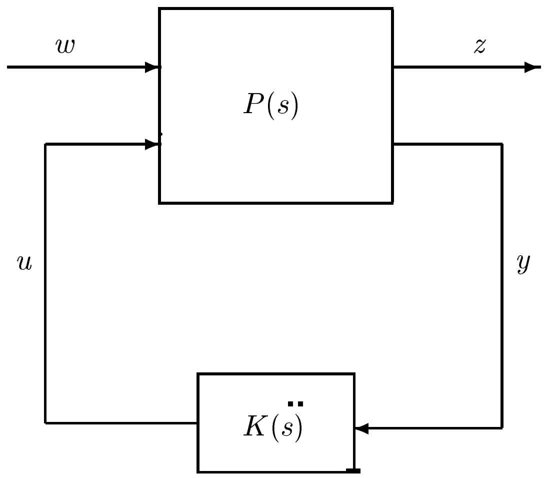

Figure 4: The standard $\mathcal{H}_\infty$ configuration

$$=: \quad \mathcal{F}_l(P,K)w$$

where $\mathcal{F}_l(P,K)$ is called the *lower linear fractional transformation* of P and K. The design objective now becomes

$$\min_{K\,stabilizing} \|\mathcal{F}_l(P,K)\|_\infty \tag{3.4}$$

and is referred to as the $\mathcal{H}_\infty$ *optimisation problem.*

Referring to the problem in (3.3), it is easy to derive its standard form by defining $w = r$, $z = \begin{bmatrix} e \\ u \end{bmatrix}$, $y = e$ and $u = u$. Consequently, the interconnected system

$$P = \begin{bmatrix} I & -G \\ 0 & I \\ I & -G \end{bmatrix}$$

where we may set

$$\begin{aligned} P_{11} &= \begin{bmatrix} I \\ 0 \end{bmatrix} & \qquad P_{12} = \begin{bmatrix} -G \\ I \end{bmatrix} \\ P_{21} &= I & \qquad P_{22} = -G. \end{aligned}$$

4 $\mathcal{H}_\infty$ Sub-Optimal Solutions

The solution to the optimisation problem (3.4) is not unique except in the scalar case. Generally speaking, there are no analytic formulae for the solutions. In practical design, it is usually sufficient to find a stabilizing controller K such that the $\mathcal{H}_\infty$-norm of the closed-loop transfer function is less than a given positive number, i.e.,

$$\|\mathcal{F}_l(P,K)\|_\infty < \gamma, \tag{4.1}$$

where $\gamma > \gamma_o := \min_{K stabilizing} \|\mathcal{F}_l(P,K)\|_\infty$. This is called the $\mathcal{H}_\infty$ *sub-optimal problem*. When certain conditions are satisfied, there are formulae to construct a set of controllers which solve the problem (4.1). The solution set is charaterized by a free parameter $Q(s)$, which is stable and of ∞-norm less than γ.

It is imaginable that if we successively reduce the value of γ, starting from a relatively big number to ensure the existence of a sub-optimal solution, we may obtain an optimal solution. It should, however, be pointed out here that when γ is approaching its minimum value γ_o the problem would become more and more ill-conditioned numerically. Hence, the 'solution' thus obtained might be very unreliable.

4.1 Solution formulae for normalized systems

Let the state-space description of the generalized (interconnected) system P in Figure 4 be given by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1 w(t) + B_2 u(t) \\ z(t) &= C_1 x(t) + D_{11} w(t) + D_{12} u(t) \\ y(t) &= C_2 x(t) + D_{21} w(t) \end{aligned}$$

where $x(t) \in R^n$ is the state vector, $w(t) \in R^{m_1}$ the exogeneous input vector, $u(t) \in R^{m_2}$ the control input vector, $z(t) \in R^{p_1}$ the error (output) vector, and $y(t) \in R^{p_2}$ the measurement vector, with $p_1 \geq m_2$ and $p_2 \leq m_1$. $P(s)$ may be further denoted as

$$\begin{aligned} P(s) &= \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} \\ &= \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right] \end{aligned} \tag{4.2}$$

$$=: \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

Note that in the above definition it is assumed that there is no direct link between the control input and the measurement output, i.e., $D_{22} = 0$. This assumption is reasonable because most industrial control systems are strictly proper and the corresponding $P(s)$ would have a zero D_{22} in a sensible design configuration. The case of non-zero direct term between $u(t)$ and $y(t)$ will be however considered in §4.3 for the sake of completeness.

The $\mathcal{H}_\infty$ solution formulae use solutions of two algebraic Riccati equations (ARE). An algebraic Riccati equation

$$E^T X + XE - XWX + Q = 0,$$

where $W = W^T$ and $Q = Q^T$, uniquely corresponds to a Hamiltonian matrix $\begin{bmatrix} E & -W \\ -Q & -E^T \end{bmatrix}$. The stabilizing solution X, if it exists, is a symmetric matrix which solves the ARE and is such that $E - WX$ is a stable matrix. The stabilizing solution is denoted as

$$X := \mathbf{Ric} \begin{bmatrix} E & -W \\ -Q & -E^T \end{bmatrix}.$$

Define

$$R_n := {D_{1*}}^T D_{1*} - \begin{bmatrix} \gamma^2 I_{m_1} & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$\tilde{R}_n := D_{*1} D_{*1}^T - \begin{bmatrix} \gamma^2 I_{p_1} & 0 \\ 0 & 0 \end{bmatrix}$$

where

$$D_{1*} = \begin{bmatrix} D_{11} & D_{12} \end{bmatrix} \quad \text{and} \quad D_{*1} = \begin{bmatrix} D_{11} \\ D_{21} \end{bmatrix}.$$

Assume that R_n abd $\tilde{R}_n$ are non-singular. We define two Hamiltonian matrices $\mathbf{H}$ and $\mathbf{J}$ as

$$\mathbf{H} := \begin{bmatrix} A & 0 \\ -C_1^T C_1 & -A^T \end{bmatrix} - \begin{bmatrix} B \\ -C_1^T D_{1*} \end{bmatrix} R_n^{-1} \begin{bmatrix} D_{1*}^T C_1 & B^T \end{bmatrix}$$

$$\mathbf{J} := \begin{bmatrix} A^T & 0 \\ -B_1B_1^T & -A \end{bmatrix} - \begin{bmatrix} C^T \\ -B_1D_{*1}^T \end{bmatrix} \tilde{R}_n^{-1} \begin{bmatrix} D_{*1}B_1^T & C \end{bmatrix}$$

Let

$$X := \mathbf{Ric\ (H)}$$
$$Y := \mathbf{Ric\ (J)}.$$

Based on X and Y, a state feedback matrix F and an observer gain matrix L can be constructed, which will be used in the solution formulae,

$$\begin{aligned} F &:= -R_n^{-1}({D_{1*}}^TC_1 + B^TX) =: \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} =: \begin{bmatrix} F_{11} \\ F_{12} \\ F_2 \end{bmatrix} \\ L &:= -(B_1{D_{*1}}^T + YC^T)\tilde{R}_n^{-1} =: \begin{bmatrix} L_1 & L_2 \end{bmatrix} =: \begin{bmatrix} L_{11} & L_{12} & L_2 \end{bmatrix} \end{aligned}$$

where $F_1,\ F_2,\ F_{11}$ and F_{12} are of $m_1, m_2, m_1 - p_2$ and p_2 rows, respectively, and $L_1,\ L_2,\ L_{11}$ and L_{12} of $p_1, p_2, p_1 - m_2$ and m_2 columns, respectively.

Glover and Doyle [11] derived necessary and sufficient conditions for the existence of an $\mathcal{H}_\infty$ sub-optimal solution and further parametrized all such controllers. The results are obtained under the following assumptions.

A1 $(A,\ B_2)$ is stabilizable and $(C_2,\ A)$ detectable;

A2 $D_{12} = \begin{bmatrix} 0 \\ I_{m_2} \end{bmatrix}$ and $D_{21} = \begin{bmatrix} 0 & I_{p_2} \end{bmatrix}$;

A3 $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank for all ω;

A4 $\begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix}$ has full row rank for all ω.

Together with appropriate partition of $D_{11} = \begin{bmatrix} D_{1111} & D_{1112} \\ D_{1121} & D_{1122} \end{bmatrix}$, where D_{1122} has m_2 rows and p_2 columns, the solution formulae are given in the following theorem.

Theorem 2 [8] *Suppose $P(s)$ satisfies the assumptions* $\mathbf{A1} - \mathbf{A4}$.
(a) There exists an internally stabilizing controller $K(s)$ such that $\|\mathcal{F}_l(P,\ K)\|_\infty < \gamma$ if and only if
(i)

$$\gamma > max(\bar{\sigma}[D_{1111}, D_{1112}], \bar{\sigma}[{D_{1111}}^T, {D_{1121}}^T])$$

and
(ii) there exist stabilizing solutions $X \geq 0$ and $Y \geq 0$ satisfying the two AREs corresponding to the Hamiltonian matrices $\mathbf{H}$ and $\mathbf{J}$, respectively, and such that

$$\rho(XY) < \gamma^2$$

where $\rho(\cdot)$ denotes the spectral radius.

(b) Given that the conditions of part (a) are satisfied, then all rational, internally stabilizing controllers, $K(s)$, satisfying $\|\mathcal{F}_l(P,\ K)\|_\infty < \gamma$ are given by

$$K(s) = \mathcal{F}_l(M, \Phi)$$

for any rational $\Phi(s) \in \mathcal{H}_\infty$ such that $\|\Phi(s)\|_\infty < \gamma$, where $M(s)$ has the realization

$$M(s) = \left[\begin{array}{c|cc} \hat{A} & \hat{B}_1 & \hat{B}_2 \\ \hline \hat{C}_1 & \hat{D}_{11} & \hat{D}_{12} \\ \hat{C}_2 & \hat{D}_{21} & 0 \end{array}\right]$$

and

$$\hat{D}_{11} = -D_{1121}{D_{1111}}^T(\gamma^2 I - D_{1111}{D_{1111}}^T)^{-1}D_{1112} - D_{1122}.$$

$\hat{D}_{12} \in R^{m_2 \times m_2}$ *and* $\hat{D}_{21} \in R^{p_2 \times p_2}$ *are any matrices (e.g. Cholesky factors) satisfying*

$$\hat{D}_{12}\hat{D}_{12}^T = I - D_{1121}(\gamma^2 I - {D_{1111}}^T D_{1111})^{-1}{D_{1121}}^T$$

$$\hat{D}_{21}^T\hat{D}_{21} = I - {D_{1112}}^T(\gamma^2 I - D_{1111}{D_{1111}}^T)^{-1}D_{1112}$$

and

$$\hat{B}_2 = Z(B_2 + L_{12})\hat{D}_{12}$$

$$
\begin{aligned}
\hat{C}_2 &= -\hat{D}_{21}(C_2 + F_{12}) \\
\\
\hat{B}_1 &= -ZL_2 + \hat{B}_2\hat{D}_{12}^{-1}\hat{D}_{11} \\
&= -ZL_2 + Z(B_2 + L_{12})\hat{D}_{11} \\
\\
\hat{C}_1 &= F_2 + \hat{D}_{11}\hat{D}_{21}^{-1}\hat{C}_2 \\
&= F_2 - \hat{D}_{11}(C_2 + F_{12}) \\
\\
\hat{A} &= A + BF + \hat{B}_1\hat{D}_{21}^{-1}\hat{C}_2 \\
&= A + BF - \hat{B}_1(C_2 + F_{12})
\end{aligned}
$$

where

$$Z = (I - \gamma^{-2}YX)^{-1}.$$

■

When $\Phi(s) = 0$ is chosen, the corresponding sub-optimal controller is called the *central* controller which is widely used in the $\mathcal{H}_\infty$ optimal design and has the state-space form

$$K(s) = \left[\begin{array}{c|c} \hat{A} & \hat{B}_1 \\ \hline \hat{C}_1 & \hat{D}_{11} \end{array}\right]$$

In the assumptions made earlier, **A2** assumes that the matrices D_{12} and D_{21} are in normalized forms and the system $P(s)$ is thus so called a normalized system. The case that those two matrices are of full rank but not necessarily in the normalized forms will be discussed next.

4.2 Normalization transformations

Detailed discussion on the normalization can be found in the *Niconet* report [2], where solution formulae directly from the original system data are presented with various cases of dimensions.

Assume that D_{12} is of full column rank and D_{21} of full row rank, respectively, but not already in the normalized forms. There exist orthonormal matrices U_{12}, V_{12}, U_{21} and V_{21}, using SVD or otherwise, such that

$$U_{12}D_{12}V_{12}^T = \begin{bmatrix} 0 \\ \Sigma_{12} \end{bmatrix}$$

$$U_{21}D_{21}V_{21}^T = \begin{bmatrix} 0 & \Sigma_{21} \end{bmatrix}$$

where $\Sigma_{12} : m_2 \times m_2$ and $\Sigma_{21} : p_2 \times p_2$ are nonsingular. Furthermore, we have

$$\begin{aligned} U_{12}D_{12}V_{12}^T\Sigma_{12}^{-1} &= \begin{bmatrix} 0 \\ I \end{bmatrix} \\ \Sigma_{21}^{-1}U_{21}D_{21}V_{21}^T &= \begin{bmatrix} 0 & I \end{bmatrix} \end{aligned}$$

The right hand sides of the above equations are now in the normalised form.

When $p_1 > m_2$ and $p_2 < m_1$, the matrices U_{12} and V_{21} can be partitioned as

$$\begin{aligned} U_{12} &= \begin{bmatrix} U_{121} \\ U_{122} \end{bmatrix} \\ V_{21} &= \begin{bmatrix} V_{211} \\ V_{212} \end{bmatrix} \end{aligned}$$

with $U_{121} : (p_1 - m_2) \times p_1, U_{122} : m_2 \times p_1, V_{211} : (m_1 - p_2) \times m_1$ and $V_{212} : p_2 \times m_1$.

The normalisation of $P(s)$ into $\bar{P}(s)$ is based on the above transformations and shown in Figure 5.

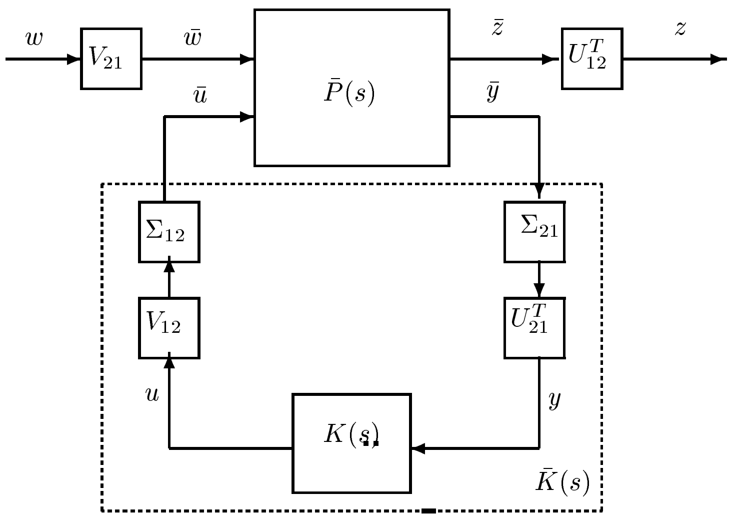

Figure 5: Normalization configuration

Given $P(s)$, the state-space form of $\bar{P}(s)$ is obtained as follows

$$\bar{B}_1 = B_1V_{21}^T$$

$$
\begin{aligned}
\bar{B}_2 &= B_2 V_{12}^T \Sigma_{12}^{-1} \\
\bar{C}_1 &= U_{12} C_1 \\
\bar{C}_2 &= \Sigma_{21}^{-1} U_{21} C_2 \\
\bar{D}_{11} &= U_{12} D_{11} V_{21}^T = \begin{bmatrix} U_{121} D_{11} V_{211}^T & U_{121} D_{11} V_{212}^T \\ U_{122} D_{11} V_{211}^T & U_{122} D_{11} V_{212}^T \end{bmatrix} \\
\bar{D}_{12} &= \begin{bmatrix} 0 \\ I \end{bmatrix} = U_{12} D_{12} V_{12}^T \Sigma_{12}^{-1} \\
\bar{D}_{21} &= \begin{bmatrix} 0 & I \end{bmatrix} = \Sigma_{21}^{-1} U_{21} D_{21} V_{21}^T.
\end{aligned}
$$

Since V_{21} and U_{12} are orthonormal, $||T_{zw}||_\infty = ||T_{\bar{z}\bar{w}}||_\infty$, with obviously $K(s) = V_{12}^T \Sigma_{12}^{-1} \bar{K}(s) \Sigma_{21}^{-1} U_{21}$, where $\bar{K}(s)$ is a sub-optimal solution with regard to $\bar{P}(s)$.

4.3 The case of $D_{22} \neq 0$

When there is a direct link between the control input and the measurement output, the matrix D_{22} will not disappear in (4.2). The controller formulae for the case $D_{22} \neq 0$ are discussed in this section.

As a matter of fact, the D_{22} term can be easily separated from the rest of the system as depicted in Figure 6. A controller $K(s)$ for the system without D_{22} will be synthesized first, and then the controller $\widetilde{K}(s)$ for the original system can be recovered from $K(s)$ and D_{22} by

$$
\widetilde{K}(s) = K(s)(I + D_{22} K(s))^{-1}
$$

The state-space model of $\tilde{K}(s)$ can be derived as

$$
\tilde{K}(s) = \left[\begin{array}{c|c} A_K - B_K D_{22}(I + D_K D_{22})^{-1} C_K & B_K (I + D_{22} D_K)^{-1} \\ \hline (I + D_K D_{22})^{-1} C_K & D_K (I + D_{22} D_K)^{-1} \end{array} \right]
$$

where we assume that

$$
K(s) = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array} \right]
$$

5 Conclusions

This report briefly introduces the $\mathcal{H}_\infty$ optimisation approach and, hopefully, would provide some general ideas to the user who wants to use the subroutines in the *SLICOT* to synthesize an $\mathcal{H}_\infty$ controller. The relevant subroutines are fully described in the *Niconet* report [1].

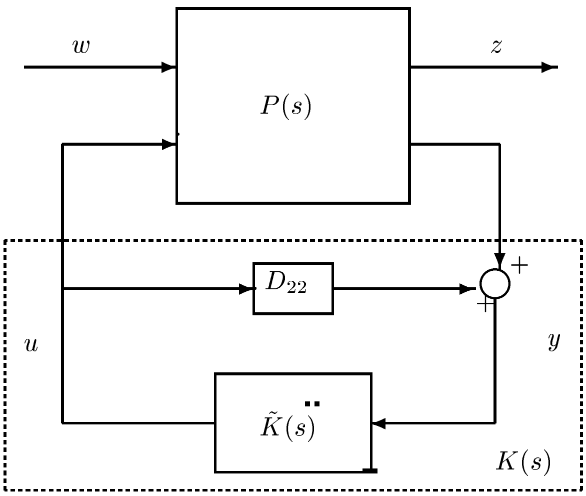

Figure 6: The case of non-zero D_{22}

The standard $\mathcal{H}_\infty$ optimisation problem, as discussed in this report, would yield good robust stabilization design and good performance with regard to the nominal plant. It would not, however, in general produce a controller which leads to a closed-loop system meeting the performance specifications even in the presence of dynamic perturbations, i.e., not achieving the robust performances. A more powerful robust design approach which considers the robust performance is the μ-synthesis/analysis method [12].

Another related robust design method is the $\mathcal{H}_\infty$ loop shaping design procedure (LDSP) [9]. From our experience, the $\mathcal{H}_\infty$ LSDP is more effective in dealing with many industrial robust design problems.

It should also note that to successfully design a control system depends heavily on the understanding of the underlying plant as well as the familiarization of the design approach and equipped with effective and efficient computation tools. Many difficult and subtle problems would emerge in applying the $\mathcal{H}_\infty$ optimisation approach. The report [3] addresses some of them. It is also our belief that numerically advanced subroutines in the *SLICOT* would help achieve more reliable design.

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